SPECTRAL THEORY OF DISSIPATIVE $q$-STURM–LIOUVILLE PROBLEMS

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Abstract

This paper is devoted to studying a $q$-analogue of Sturm–Liouville operators. We formulate a dissipative $q$-difference operator in a Hilbert space. We construct a self adjoint dilation of such operators. We also construct a functional model of the maximal dissipative operator which is based on the method of Pavlov and define its characteristic function. Finally, we prove theorems on the completeness of the system of eigenvalues and eigenvectors of the maximal dissipative $q$-Sturm–Liouville difference operator.

1. Introduction

The $q$-difference calculus or quantum calculus initiated in the beginning of the 19th century. The subject of $q$-difference equations has evolved into a multidisciplinary subject [17]. Adıvar and Bohner investigated the eigenvalues and the spectral singularities of nonself-adjoint $q$-difference equations of second order with spectral singularities in [1]. Huseynov and Bairamov examined of the properties of eigenvalues and eigenvectors of a quadratic pencil of $q$-difference equations in [20]. In [31], Shi and Wu presented several classes of explicit self-adjoint Sturm–Liouville difference operators with either a non-Hermitian leading coefficient function, or a non-Hermitian potential function, or a non-definite weight function, or a non-self-adjoint bound-

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In [8], Annaby and Mansour studied a $q$-analogue of Sturm–Liouville eigenvalue problems and formulated a self-adjoint $q$-difference operator in a Hilbert space. They also discussed properties of the eigenvalues and the eigenfunctions. Annaby et al. established the $q$-Titchmarsh–Weyl theory for singular $q$-Sturm–Liouville problems and defined $q$-limit-point and $q$-limit circle singularities and gave sufficient conditions which guarantee that the singular point is in a limit point case in [9]. Eryilmaz investigated $q$-Sturm–Liouville boundary value problem in the Hilbert space with a spectral parameter in the boundary condition in [18]. One can consult the reference [21] for some definitions and theorems on $q$-derivative, $q$-integration, $q$-exponential function, $q$-trigonometric function, $q$-Taylor formula, $q$-Beta and Gamma functions, Euler-Maclaurin formula, etc. The spectral analysis of nonself-adjoint (dissipative) operators is based on ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. Using a functional model of a nonself-adjoint operator as a principal tool, spectral properties of such operators are investigated. The functional model of nonself-adjoint dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other disciplines. The construction of functional models for dissipative operators, natural analogues of spectral decompositions for self-adjoint operators is based on Sz. Nagy–Foias dilation theory [25] and Lax–Phillips scattering theory [24]. Pavlov’s approach ([27]–[29]) to the model construction of dissipative extensions of symmetric operators was followed by B. Allahverdiev in his works [2]–[7] and some others, and by the group of authors [10]–[12], where the theory of the dissipative Schrödinger operator on a finite interval was applied to the problems arising in the semiconductor physics. In [13]–[16], Pavlov’s functional model was extended to (general) dissipative operators which are finite dimensional extensions of a symmetric operator, and the corresponding dissipative and Lax–Phillips scattering problems were investigated in some detail. This method has already been used in many papers [2]–[7], [30]. In this paper, we apply this method to $q$-Sturm–Liouville operators in space $L^2_q(0,a)$. The organization of this document is as follows: In Section 2, we constructed a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, self-adjoint and other extensions of dissipative $q$-difference operators of the Sturm–Liouville type in terms of boundary conditions. Furthermore, we constructed a self-adjoint dilation of the dissipative $q$-difference operators of the Sturm–Liouville type. We presented its incoming and outgoing spectral representations which makes it possible to determine the scattering matrix of the dilation according to the Lax and Phillips scheme [24] in Section 3. Later, a functional model of the dissipative $q$-difference operators of the Sturm–Liouville type is constructed by methods of Pavlov [27]–[29] and defined its characteristic functions. Finally, we prove a theorem on completeness of the system of eigenvectors and associated vectors of dissipative
operators under consideration in Section 4. While proving our results, we use the machinery of [2]–[7].

2. Self-adjoint dilation of dissipative \(q\)-difference operators of the Sturm–Liouville type

Following the standard notations in [21], let \(q\) be a positive number with \(0 < q < 1\), \(A \subset \mathbb{R}\) and \(a \in \mathbb{C}\). A \(q\)-difference equation is an equation that contains \(q\)-derivatives of a function defined on \(A\). Let \(y(x)\) be a complex-valued function on \(A\). The \(q\)-difference operator \(D_q\) is defined by

\[
D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \quad \text{for all } x \in A.
\]

where \(\mu(x) = (q - 1)x\). The \(q\)-derivative at zero is defined by

\[
D_q y(0) = \lim_{n \to \infty} \frac{y(q^n x) - y(0)}{q^n x}, \quad x \in A,
\]

if the limit exists and does not depend on \(x\). A right inverse to \(D_q\), the Jackson \(q\)-integration is given by

\[
\int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in A,
\]

provided that the series converges, and

\[
\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in A.
\]

Let \(L^2_q(0, a)\) be the space of all complex-valued functions defined on \([0, a]\) such that

\[
\|f\| := \left( \int_0^a |f(x)|^2 d_q x \right)^{1/2} < \infty.
\]

The space \(L^2_q(0, a)\) is a separable Hilbert space with the inner product

\[
(f, g) := \int_0^a f(x) \overline{g(x)} d_q x, \quad f, g \in L^2_q(0, a).
\]
We will consider the basic Sturm–Liouville operator

\[ l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x), \quad 0 \leq x \leq a < +\infty \]

where \( v(x) \) is defined on \([0, a]\) and continuous at zero. The \( q \)-Wronskian of \( y_1(x), y_2(x) \) is defined to be

\[ W_q(y_1, y_2)(x) := y_1(x) D_q y_2(x) - y_2(x) D_q y_1(x), \quad x \in [0, a]. \]

Let us define

\[ D := \{ y(x) \in L^2_q(0, a) : y(x) \text{ and } D_q y(x) \text{ are continuous in } [0, a) \} \]

and

\[ D_0 := \{ y(x) \in D : y(0) = y(a) = D_{q^{-1}} y(0) = D_{q^{-1}} y(a) = 0 \}. \]

Let \( L_0 \) denote the closure of the minimal operator generated by (2.1) and \( D_0 \) be its domain. Besides, we denote the maximal operator by \( L \), where \( D \) is the domain of \( L \). Furthermore \( L = L^*_0 \) [26]. Suppose that the operator \( L_0 \) has defect index \((2, 2)\), so the case of Weyl’s limit-circle occurs for \( l \).

For every \( y, z \in D \) we have \( q \)-Lagrange’s identity (see [8])

\[ (Ly, z) - (y, Lz) = [y, \bar{z}](a) - [y, \bar{z}](0) \]

where \([y, \bar{z}] := y(x) \overline{D_{q^{-1}} z(x) - D_{q^{-1}} y(x) \bar{z}(x)}\).

Let's define by \( \Gamma_1, \Gamma_2 \) the linear maps from \( D \) to \( \mathbb{C}^2 \) by the formulae

\[
\Gamma_1 f = \begin{pmatrix} -y(0) \\ y(a) \end{pmatrix}, \quad \Gamma_2 f = \begin{pmatrix} D_{q^{-1}} y(0) \\ D_{q^{-1}} y(a) \end{pmatrix}, \quad y \in D.
\]

For any \( y, z \in D \), we have

\[
(Ly, z) - (y, Lz) = [y, \bar{z}](a) - [y, \bar{z}](0)
= (\Gamma_1 f, \Gamma_2 g)_{\mathbb{C}^2} - (\Gamma_2 f, \Gamma_1 g)_{\mathbb{C}^2}.
\]

**Theorem 1.** The triple \((\mathbb{C}^2, \Gamma_1, \Gamma_2)\) defined by (2.2) is a boundary spaces of the operator \( L_0 \).

**Proof.** The proof is obtained from (2.3) and definition of the boundary value space. \(\square\)
Recall that a linear operator \( T \) (with dense domain \( D(T) \)) acting on some Hilbert space \( H \) is called dissipative (accretive) if \( \text{Im}(Tf,f) \geq 0 \) (\( \text{Im}(Tf,f) \leq 0 \)) for all \( f \in D(T) \) and maximal dissipative (maximal accretive) if it does not have a proper dissipative (accretive) extension. From [19, 22], following theorem is obtained.

**Theorem 2.** For any contraction \( K \) in \( C^2 \) the restriction of the operator \( L \) to the set of functions \( y \in D \) satisfying either

\[
(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0
\]

or

\[
(K - I)\Gamma_1 y - i(K + I)\Gamma_2 y = 0
\]

is respectively the maximal dissipative and accretive extension of the operator \( L_0 \). Conversely, every maximal dissipative (accretive) extension of the operator \( L_0 \) is the restriction of \( L \) to the set of functions \( y \in D \) satisfying (2.4) ((2.5)), and the extension uniquely determines the contraction \( K \). Conditions (2.4) ((2.5)), in which \( K \) is an isometry describe the maximal symmetric extensions of \( L_0 \) in \( L^2_a(0,a) \). If \( K \) is unitary, these conditions define self-adjoint extensions.

In particular, the boundary conditions

\[
y(0) + h_1 D_{q-1} y(0) = 0
\]

\[
y(a) + h_2 D_{q-1} y(a) = 0
\]

with \( \text{Im} h_1 \geq 0 \) or \( h_1 = \infty \), \( \text{Im} h_2 \geq 0 \) or \( h_2 = \infty \), describe the maximal dissipative (self-adjoint) extensions of \( L_0 \) with separated boundary conditions.

In the sequel we shall study the maximal dissipative operator \( L_K \), where \( K \) is the strict contraction in \( C^2 \) generated by the expression \( l(y) \) and boundary condition (2.4). Since \( K \) is a strict contraction, the operator \( K + I \) must be invertible, and the boundary condition (2.4) is equivalent to the condition

\[
\Gamma_2 y + G \Gamma_1 y = 0,
\]

where \( G = -i(K + I)^{-1}(K - I) \), \( \text{Im} G > 0 \), and \( -K \) is the Cayley transform of the dissipative operator \( G \). We denote \( \tilde{L}_G \) (\( = L_K \)) the dissipative operator generated by the expression \( l(y) \) and boundary condition (2.8).

Let

\[
G = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}
\]
where $\text{Im} \ h_1 > 0$, $\text{Im} \ h_2 > 0$ and $C^2 = 2 \text{Im} \ G$, $C > 0$. Then the boundary condition (2.8) coincides with the separated boundary conditions (2.6) and (2.7).

Let us add the “incoming” and “outgoing” subspaces

$$D_- = L^2((-\infty, 0); \mathbb{C}^2) \quad \text{and} \quad D_+ = L^2((0, \infty); \mathbb{C}^2)$$

to $H = L^2_2(0, a)$. The orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$ is called main Hilbert space of the dilation.

In the space $\mathcal{H}$, we consider the operator $\mathcal{L}_G$ on the set $D(\mathcal{L}_G)$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\langle \varphi_-, y, \varphi_+ \rangle = \left\langle \frac{d\varphi_-}{d\xi}, l(y), i \frac{d\varphi_+}{d\xi} \right\rangle$$

satisfying the conditions: $\varphi_- \in W^1_2((-\infty, 0); \mathbb{C}^2)$, $\varphi_+ \in W^1_2((0, \infty); \mathbb{C}^2)$, $\hat{y} \in H$,

$$\Gamma_2 \hat{y} + \Gamma_1 = C \varphi_- (0), \quad \Gamma_2 \hat{y} + G^* \Gamma_1 = C \varphi_+ (0),$$

where $W^1_2$ are Sobolev spaces and $C^2 := 2 \text{Im} \ G$, $C > 0$.

**Theorem 3.** The operator $\mathcal{L}_G$ is self-adjoint in $\mathcal{H}$ and it is a self-adjoint dilation of the operator $\mathcal{L}_G$ (= $L_K$).

**Proof.** We first prove that $\mathcal{L}_G$ is symmetric in $\mathcal{H}$. Namely $(\mathcal{L}_G f, g)_\mathcal{H} = (f, \mathcal{L}_G g)_\mathcal{H} = 0$. Let $f, g \in D(\mathcal{L}_G)$, $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Then we have

$$(\mathcal{L}_G f, g)_\mathcal{H} = (f, \mathcal{L}_G g)_\mathcal{H}$$

$$= (\mathcal{L} \langle \varphi_-, y, \varphi_+ \rangle, \langle \psi_-, z, \psi_+ \rangle) - (\langle \varphi_-, y, \varphi_+ \rangle, \mathcal{L} \langle \psi_-, z, \psi_+ \rangle)$$

$$= i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2} + (l(y), z) \rangle_{\mathbb{H}} - (y, l(z)) \rangle_{\mathbb{H}} - i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2}$$

$$= i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2} - i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2} + [\hat{y}, \bar{\varphi}_0] (a) - [\hat{y}, \bar{\varphi}_0] (0)$$

We obtain by direct computation

$$i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2} - i \varphi_0 (0), \psi_0 (0) \rangle_{\mathbb{C}^2} + [y, \bar{\varphi}_0] (a) - [y, \bar{\varphi}_0] (0) = 0.$$
By choosing elements with suitable components as the \( f \in D(L_G) \) in (2.10), one can show that \( \psi_\pm \in W^1_2((\infty,0);\mathbb{C}^2) \), \( \psi_+ \in W^1_2((0,\infty);\mathbb{C}^2) \), \( g \in D(L) \) and \( g^* = L_g \), where operator \( L \) is defined (2.9). Therefore (2.10) is obtained from \((L_f,g)_H = (f,L_g)_H\) for all \( f \in D(L_G^*) \). Furthermore, \( g \in D(L_G^*) \) satisfies the following conditions

\[
\Gamma_2 z + G \Gamma_1 z = C \varphi_-(0),
\]

\[
\Gamma_2 z + G^* \Gamma_1 z = C \varphi_+(0).
\]

Hence, \( D(L_G^*) \subseteq D(L_G) \), i.e., \( L_G = L_G^* \).

The self-adjoint operator \( L_G \) generates on \( H \) a unitary group \( U_t = \exp(iL_G t) \) \((t \in \mathbb{R}_+ = (0,\infty))\). Let denote by \( P : H \rightarrow H \) and \( P_1 : H \rightarrow H \) the mapping acting according to the formulae \( P : \langle \varphi_-,\varphi_+ \rangle \rightarrow y \) and \( P_1 : y \rightarrow (0,y,0) \). Let \( Z_t := PU_tP_1, \ t \geq 0 \). The family \( \{Z_t\} \ (t \geq 0) \) of operators is a strongly continuous semigroup of completely non-unitary contraction on \( H \). Let us denote by \( B_G \) the generator of this semigroup: \( B_G y = \lim_{t \to 1} \left( e^{it1} - 1 \right)(Z_1 y - y) \). The domain of \( B_G \) consists of all the vectors for which the limit exists. The operator \( B_G \) is dissipative. The operator \( L_G \) is called the self-adjoint dilation of \( B_G \) (see [6, 23, 25]). We should denote \( B_G \) by \( \tilde{L}_G \) and then it is shown that \( L_G \) is self-adjoint dilation of \( B_G \). To show this, it is sufficient to verify the equality

\[
P(L_G - \lambda I)^{-1}P_1 y = \left( \tilde{L}_G - \lambda I \right)^{-1} y, y \in H, \quad \text{Im} \ h < 0.
\]

For this purpose, we substitute \( (L_G - \lambda I)^{-1}P_1 y = g = \langle \psi_-,\tilde{z},\psi_+ \rangle \) implies that \( (L_G - \lambda I)g = P_1 y \), and hence \( \{l(z) - \lambda z = y, \ \psi_-(\xi) = \psi_-(0)e^{-i\lambda \xi} \text{ and } \psi_+(\xi) = \psi_+(0)e^{i\lambda \xi} \} \). Since \( g \in D(L_G) \), \( \psi_+ \in W^1_2((\infty,0);\mathbb{C}^2) \), it follows that \( \psi_-(0) = 0 \), and consequently \( z \) satisfies the boundary condition \( \Gamma_2 z + G \Gamma_1 z = 0 \). Therefore \( z \in D(\tilde{L}_G) \), and since point \( \lambda \) with \( \text{Im} \lambda < 0 \) cannot be an eigenvalue of dissipative operator, then \( z = (\tilde{L}_G - \lambda I)^{-1} y \). Thus

\[
(L_G - \lambda I)^{-1}P_1 y = \langle 0, (\tilde{L}_G - \lambda I)^{-1} \tilde{y}, C^{-1}(\Gamma_2 y + G^* \Gamma_1 y)e^{-i\lambda \xi} \rangle
\]

for \( y \in H \) and \( \text{Im} \lambda < 0 \). By applying the mapping \( P \), we obtain (2.11), and

\[
(L_G - \lambda I)^{-1} = P(L_G - \lambda I)^{-1}P_1 = -iP \int_0^\infty U_t e^{-i\lambda t} dt P_1
\]

\[
= -i \int_0^\infty Z_t e^{-i\lambda t} dt = (B_G - \lambda I)^{-1}, \quad \text{Im} \lambda < 0,
\]
so this clearly shows that $\tilde{L}_G = B_G$. □

3. Functional model of dissipative $q$-Sturm–Liouville operators

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax–Phillips [24] i.e., it has orthogonal incoming and outgoing subspaces $D_- = \langle L^2((-\infty, 0); C^2), 0, 0 \rangle$ and $D_+ = \langle 0, 0, L^2((0, \infty); C^2) \rangle$ having the following properties:

(i) $U_t D_- \subset D_-$, $t \leq 0$ and $U_t D_+ \subset D_+$, $t \geq 0$;
(ii) $\bigcup_{t \leq 0} U_t D_- = \bigcup_{t \geq 0} U_t D_+ = \{0\}$;
(iii) $\bigcup_{t \leq 0} U_t D_- = \bigcup_{t \geq 0} U_t D_+ = \mathcal{H}$;
(iv) $D_- \perp D_+$.

To prove property (i) for $D_+$ (the proof for $D_-$ is similar), let us set $R_\lambda = (L^G - \lambda I)^{-1}$. For all $\lambda$, with $\text{Im} \lambda < 0$ and for any $f = \langle 0, 0, \varphi_+ \rangle \in D_+$, as $R_\lambda f \in D_+$, we have

$$R_\lambda f = \left\langle 0, 0, -ie^{-\lambda t} \int_0^\xi e^{i\lambda s} \varphi_+(s) \, ds \right\rangle.$$

Therefore, if $g \perp D_+$, then

$$0 = (R_\lambda f, g)_\mathcal{H} = -i \int_0^\infty e^{-i\lambda t}(U_t f, g)_\mathcal{H} \, dt, \quad \text{Im} \lambda < 0.$$

which implies that $(U_t f, g)_\mathcal{H} = 0$ for all $t \geq 0$. Hence, for $t \geq 0, U_t D_+ \subset D_+$, so property (i) has been proved.

Now, to prove property (ii), we firstly define the mappings $P^+: \mathcal{H} \to L^2((0, \infty); C^2)$ and $P^+_1: L^2((0, \infty); C^2) \to D_+$ as follows $P^+: \langle \varphi_-, \tilde{g}, \varphi_+ \rangle \to \varphi_+$ and $P^+_1: \varphi \to (0, 0, \varphi)$, respectively. We take into consider that the semigroup of isometries $U^+_t := P^+ U_t P^+_1 (t \geq 0)$ is a one-sided shift in $L^2((0, \infty); C^2)$. Indeed, the generator of the semigroup of the one-sided shift $V_t$ in $L^2((0, \infty); C^2)$ is the differential operator $i \left( \frac{d}{dt} \right)$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator $S$ of the semigroup of isometries $U^+_t (t \geq 0)$ is the operator $S\varphi = P^+ L^G P^+_1 \varphi = P^+ L^G \langle 0, 0, \varphi \rangle = P^+ \langle 0, 0, i \left( \frac{d}{dt} \right) \varphi \rangle = i \left( \frac{d}{dt} \right) \varphi$, where $\varphi \in W^2_1((0, \infty); C^2)$ and $\varphi(0) = 0$. Since
a semigroup is uniquely determined by its generator, it follows that \( U_t^+ = V_t \), and, hence,

\[
\bigcap_{t \geq 0} U_t D_+ = \left\{ 0, 0, \bigcap_{t \leq 0} V_t L^2((0, \infty); \mathbb{C}^2) \right\} = \{0\},
\]

so the proof of (ii) is completed.

**Definition 1.** The linear operator \( A \) with domain \( D(A) \) acting in the Hilbert space \( H \) is called completely nonselfadjoint (or simple) if there is no invariant subspace \( M \subseteq D(A) \) \( (M \neq \{0\}) \) of the operator \( A \) on which the restriction \( A \) to \( M \) is self-adjoint.

To prove property (iii) of the incoming and outgoing subspaces, we give following lemma.

**Lemma 1.** The operator \( L_G \) is completely nonself-adjoint (simple).

**Proof.** Let \( H' \subset H \) be a nontrivial subspace in which \( L_G \) induces a selfadjoint operator \( L_G' \) with domain \( D(L_G') = H' \cap D(L_G) \). If \( f \in D(L_G') \), then \( f \in D(L_{G'}) \) and

\[
0 = \frac{d}{dt} \|e^{iL_G't}f\|_H^2 = \frac{d}{dt} \langle e^{iL_G't}f, e^{iL_G't}f \rangle_H
= -2 \langle \text{Im} G \Gamma_1 e^{iL_G't}f, \Gamma_1 e^{iL_G't}f \rangle_{\mathbb{C}^2}.
\]

Consequently, we have \( \Gamma_1 e^{iL_G't}f = 0 \). For eigenvectors \( y(\lambda) \in H' \) of the operator \( L_G' \) we have \( \Gamma_1 y(\lambda) = 0 \). Using this result with boundary condition \( \Gamma_2 y + G \Gamma_1 y = 0 \), we have \( \Gamma_2 y(\lambda) = 0 \), i.e., \( y(\lambda) = 0 \). Since all solutions of \( l(y) = \lambda y \) belong to \( L_0^2(0, a) \), from this it can be concluded that the resolvent \( R_\lambda(L_G) \) is a compact operator, and the spectrum of \( L_G \) is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator \( L_G \), we obtain \( H' = \{0\} \). Hence the operator \( L_G \) is simple. The proof of Lemma 1 is completed.

Let us define \( H_\pm = \bigcup_{t \geq 0} U_t D_\pm, H_\pm = \bigcup_{t \leq 0} U_t D_\pm \).

**Lemma 2.** The equality \( H_- + H_+ = \mathcal{H} \) holds.

**Proof.** Considering property (i) of the subspace \( D_+ \), it is easy to show that the subspace \( \mathcal{H}' = \mathcal{H} \cap (H_- + H_+) \) is invariant relative to the group \( \{U_t\} \) and has the form \( \mathcal{H}' = (0, H', 0) \), where \( H' \) is a subspace in \( H \). Therefore, if the subspace \( \mathcal{H}' \) (and hence also \( H' \)) were nontrivial, then the unitary
group \(\{U'_t\}\) restricted to this subspace would be a unitary part of the group \(\{U_t\}\), and hence, the restriction \(\tilde{L}_G\) of \(L_G\) to \(H'\) would be a self-adjoint operator in \(H'\). Since the operator \(L_G\) is simple, it follows that \(H' = \{0\}\). The Lemma 2 is proved.

Assume that \(\varphi(x, \lambda)\) and \(\psi(x, \lambda)\) are solutions of \(l(y) = \lambda y\) satisfying the conditions

\[
\varphi(0, \lambda) = 0, \quad D_q\varphi(0, \lambda) = -1, \quad \psi(0, \lambda) = 1, \quad D_q\psi(0, \lambda) = 0.
\]

We denote by \(M(\lambda)\) the matrix-valued function satisfying the conditions

\[
M(\lambda)\Gamma_1\varphi = \Gamma_2\varphi, \quad M(\lambda)\Gamma_1\psi = \Gamma_2\psi.
\]

\(M(\lambda)\) is a meromorphic function on the complex plane \(\mathbb{C}\) with a countable number of poles on the real axis. Further, it is possible to show that the function \(M(\lambda)\) possesses the following properties: \(\text{Im} M(\lambda) \leq 0\) for all \(\text{Im} \lambda \neq 0\), and \(M^*(\lambda) = M(\bar{\lambda})\) for all \(\lambda \in \mathbb{C}\), except the real poles \(M(\lambda)\).

We denote by \(\theta_j(x, \lambda)\) and \(\chi_j(x, \lambda)\) (\(j = 1, 2\)) the solutions of system \(l(y) = \lambda y\) which satisfy the conditions

\[
\Gamma_1\chi_j = (M(\lambda) + G)^{-1}Ce_j, \quad \Gamma_1\theta_j = (M(\lambda) + G^*)^{-1}Ce_j, \quad (j = 1, 2),
\]

where \(e_1\) and \(e_2\) are the orthonormal basis for \(\mathbb{C}^2\).

We set

\[
U_{\lambda j}(x, \xi, \zeta) = \langle e^{-i\lambda \xi}, \chi_j(\lambda), C^{-1}(M + G^*)(M + G)^{-1}Ce^{-i\lambda \zeta}e_j \rangle, \quad (j = 1, 2).
\]

We note that the vectors \(U_{\lambda j}(x, \xi, \zeta)\) (\(j = 1, 2\)) for real \(\lambda\) do not belong to the space \(\mathcal{H}\). However, \(U_{\lambda j}(x, \xi, \zeta)\) (\(j = 1, 2\)) satisfies the equation \(\mathcal{L}U = \lambda U\) and the corresponding boundary conditions for the operator \(\mathcal{L}_h\).

By means of vector \(U_{\lambda j}(x, \xi, \zeta)\) (\(j = 1, 2\)), we define the transformation \(F_- : f \to \tilde{f}_-(\lambda)\) by formula

\[
(F_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{2} \bar{f}_j-(\lambda)e_j
\]

on the vectors \(f = \langle f_-, y, f_+ \rangle\) in which \(f_-(\xi), f_+(\zeta), y(x)\) are smooth, compactly supported functions and

\[
\tilde{f}_j-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_{\lambda j})_{\mathcal{H}}, \quad (j = 1, 2).
\]
Lemma 3. The transformation $F_-$ isometrically maps $H_-$ onto $L^2(\mathbb{R} : \mathbb{C}^2)$. For all vectors $f, g \in H_-$ the Parseval equality and the inversion formulae hold:

$$(f,g)_H = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{f}_j^-(\lambda) \overline{\tilde{g}_j^-} d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{f}_j^- (\lambda) U_{j\lambda} d\lambda,$$

where $\tilde{f}_- (\lambda) = (F_- f)(\lambda)$ and $\tilde{g}_- (\lambda) = (F_- g)(\lambda)$.

Proof. For $f, g \in D_-$,

$$f = \langle f_-, 0, 0 \rangle, \quad g = \langle g_-, 0, 0 \rangle, \quad f_-, g_- \in L^2 \left( (-\infty, 0) : \mathbb{C}^2 \right)$$

with Paley–Wiener theorem, we have

$$\tilde{f}_j^- (\lambda) = \frac{1}{\sqrt{2\pi}} (f, U_{j\lambda})_H = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (f_-(\xi), e^{-i\lambda \xi} e_j)_{\mathbb{C}^2} d\xi \in H^2_-(\mathbb{C}^2),$$

$$f_-(\xi) = \sum_{j=1}^{2} \tilde{f}_j^- (\lambda) e_j \in H^2_-(\mathbb{C}^2),$$

and by using usual Parseval equality for Fourier integrals, we have

$$(f,g)_H = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{f}_j^- (\lambda) \overline{\tilde{g}_j^-} d\lambda.$$

Here, $H^2_-(\mathbb{C}^2)$ denote the Hardy classes in $L^2(\mathbb{R} : \mathbb{C}^2)$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively.

We now extend to the Parseval equality to the whole of $H_-$. We consider in $H_-$ the dense set of $H'_-$ of the vectors obtained as follows from the smooth, compactly supported functions in $D_- : f \in H'_- \iff f = U_t f_0, \ f_0 = \langle \varphi_-, 0, 0 \rangle, \ \varphi_- \in C_0^\infty \left( (-\infty, 0) : \mathbb{C}^2 \right)$, where $t = t_f$ is a nonnegative number depending on $f$. If $f, g \in H'_-$, then for $t > t_f$ and $t > t_g$ we have $U_{-t} f, U_{-t} g \in D_-,$
moreover, the first components of these vectors belong to $C_0^\infty((-\infty,0):\mathbb{C}^2)$.
Therefore, since the operators $U_t^-(t \in \mathbb{R})$ are unitary, by the equality

$$F_- U_t^- f = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^2 (U_t^- f, U_{\chi_j}) \mathcal{H} e_j = \frac{1}{\sqrt{2\pi}} e^{i\lambda t} \sum_{j=1}^2 (f, U_{\lambda_j}^-) \mathcal{H} e_j = e^{i\lambda t} F_- f,$$

we have

$$(f,g)_{\mathcal{H}} = (U_{-t}^- f, U_{-t}^- g)_{\mathcal{H}} = (F_- U_{-t}^- f, F_- U_{-t}^- g)_{L^2}$$

$$(e^{-i\lambda t} F_- f, e^{-i\lambda t} F_- g)_{L^2} = (\tilde{f}, \tilde{g})_{L^2}.$$ 

By taking the closure (3.1), we obtain the Parseval equality for the space $H_-$. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally

$F_- H_- = \bigcup_{t \geq 0} F_- U_tD_- = \bigcup_{t \geq 0} e^{-i\lambda t} H^2(\mathbb{C}^2) = L^2(\mathbb{R}:\mathbb{C}^2),$ 

that is $F_-$ maps $H_-$ onto the whole of $L^2(\mathbb{R}:\mathbb{C}^2)$. The Lemma 3 is proved.

We set

$$U_{\chi_j}^+(x,\xi,\zeta) = \langle S_G(\lambda) e^{-i\lambda \xi} e_j, \theta_j(\lambda), e^{-i\lambda \zeta} e_j \rangle, \quad (j = 1,2),$$

where

$$(3.2) \quad S_G(\lambda) = C^{-1}(M(\lambda) + G)(M(\lambda) + G^*)^{-1}C.$$ 

We note that the vectors $U_{\chi_j}^+(x,\xi,\zeta) (j = 1,2)$ for real $\lambda$ do not belong to the space $\mathcal{H}$. However, $U_{\chi_j}^+(x,\xi,\zeta) (j = 1,2)$ satisfies the equation $LU = \lambda U$ and the corresponding boundary conditions for the operator $L_h$. With the help of vectors $U_{\chi_j}^+(x,\xi,\zeta) (j = 1,2)$, we define the transformation $F_+$:

$f \to \tilde{f}_+(\lambda)$ by $(F+f)(\lambda) := \tilde{f}_+(\lambda) := \sum_{j=1}^2 \tilde{f}_{j+}(\lambda) e_j$, $\tilde{f}_{j+} := \frac{1}{\sqrt{2\pi}}(f, U_{\chi_j}^+)_{\mathcal{H}}$

$(j = 1,2)$ on the vectors $f = \langle \varphi_-, y, \varphi_+ \rangle$ in which $\varphi_-(\xi), \varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

**Lemma 4.** The transformation $F_+$ isometrically maps $H_+$ onto $L^2(\mathbb{R}:\mathbb{C}^2)$. For all vectors $f,g \in H_+$ the Parseval equality and the inversion formula hold:

$$(f,g)_{\mathcal{H}} = (\tilde{f}_+, \tilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^2 \tilde{f}_{j+}(\lambda) \overline{\tilde{g}_{j+}(\lambda)} d\lambda,$$
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$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{f}_+(\lambda) U_{\lambda j}^+ d\lambda,$$

where $\tilde{f}_+(\lambda) = (F_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (F_+ g)(\lambda)$.

**Proof.** The proof is analogous to the Lemma 3.

It is obvious that the matrix-valued function $S_G(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (3.2), $\|S_G(\lambda)\|_{L^2} \leq 1$ for $\text{Im}\ \lambda > 0$; and $S_G(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Therefore, it explicitly follows from the formulae for the vectors $U_{\lambda j}^-$ and $U_{\lambda j}^+$ that

$$U_{\lambda j}^+ = \sum_{k=1}^{2} S_{jk}(\lambda) U_{\lambda k}^- \quad (j = 1, 2),$$

where $S_{jk}(\lambda)$ ($j = 1, 2$) are the elements of the matrix $S_G(\lambda)$. It follows from Lemmas 3 and 4 that $H_- = H_+$. Together with Lemma 2, this shows that $H_- = H_+ = \mathcal{H}$, therefore property $(iii)$ above has been proved for the incoming and outgoing subspaces. Finally property $(iv)$ is clear.

Thus, the transformation $F_-$ isometrically maps $H_-$ onto $L^2(\mathbb{R} : \mathbb{C}^2)$ with the subspace $D_-$ mapped onto $H^2_+ (\mathbb{C}^2)$ and the operators $U_t$ are transformed into the operators of multiplication by $e^{it\lambda}$. This means that $F_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $F_+$ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (3.3) that the passage from the $F_-$ representation of an element $f \in \mathcal{H}$ to its $F_+$ representation is accomplished as $\tilde{f}_-(\lambda) = S_G^{-1}(\lambda) \tilde{f}_+(\lambda)$. Consequently, according to [24] we have proved the following theorem.

**Theorem 4.** The function $S_G^{-1}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator $\mathcal{L}_G$).

Let $S(\lambda)$ be an arbitrary nonconstant inner function (see [25]) on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H^2_+ \ominus SH^2_+$. Then $K \neq \{0\}$ is a subspace of the Hilbert space $H^2_+$. We consider the semigroup of operators $Z_t$ ($t \geq 0$) acting in $K$ according to the formula $Z_t \varphi = P[e^{it\lambda} \varphi], \varphi = \varphi(\lambda) \in K$, where $P$ is the orthogonal projection from $H^2_+$ onto $K$. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T \varphi = \lim_{t \to +0} (it)^{-1}(Z_t \varphi - \varphi),$$
which $T$ is a maximal dissipative operator acting in $K$ and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator $T$ is called a model dissipative operator (we remark that this model dissipative operator, which is associated with the names of Lax–Phillips [24], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [25]. The basic assertion is that $S(\lambda)$ is the characteristic function of the operator $T$.

Let $K = (0, H, 0)$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation $F_-$ under the mapping $F_-$.

\begin{align}
\mathcal{H} &\rightarrow L^2(\mathbb{R} : \mathbb{C}^2), & f &\rightarrow \tilde{f}_- (\lambda) = (F_- f)(\lambda), \\
D_- &\rightarrow H_+^2 (\mathbb{C}^2), & D_+ &\rightarrow S_G H_+^2 (\mathbb{C}^2), \\
K &\rightarrow H_+^2 \oplus S_G H_+^2, & U_t &\rightarrow (F_- U_t F_-^{-1} \tilde{f}_-)(\lambda) = e^{i\lambda t} \tilde{f}_- (\lambda).
\end{align}

The formulas (3.4) show that operator $\tilde{L}_G$ is a unitarily equivalent to the model dissipative operator with the characteristic function $S_G(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operator coincide ([25]), we have thus proved following theorem.

**Theorem 5.** The characteristic function of the maximal dissipative operator $\tilde{L}_G$ coincides with the function $S_G(\lambda)$ defined (3.2).

4. The spectral properties of dissipative q-Sturm–Liouville

Using characteristic function, the spectral properties of the maximal dissipative operator $\tilde{L}_G(L_K)$ can be investigated. The characteristic function of the maximal dissipative operator $\tilde{L}_G$ is known to lead to information of completeness about the spectral properties of this operator. For instance, the absence of a singular factor $s(\lambda)$ of the characteristic function $S_G(\lambda)$ in the factorization $\det S_G(\lambda) = s(\lambda) B(\lambda)$ ($B(\lambda)$ is a Blaschke product) ensures completeness of the system of eigenvectors and associated vectors of the operator $\tilde{L}_G(L_K)$ in the space $L_2(0, \infty)$ (see [6, 23, 25]).

**Lemma 5.** The characteristic function $\tilde{S}_G(\lambda)$ of the operator $L_K$ has the form

\begin{align}
\tilde{S}_G(\lambda) &:= S_G(\lambda) \\
&= X_1 (I - K_1 K_1^*)^{1/2} \left( \Theta(\xi) - K_1 \right) \left( I - K_1^* \Theta(\xi) \right)^{-1} \left( I - K_1 K_1^* \right)^{1/2} X_2,
\end{align}
where $K_1 = -K$ is the Cayley transformation of the dissipative operator $G$, and $\Theta(\xi)$ is the Cayley transformation of the matrix-valued function $M(\lambda)$,

$$\xi = (\lambda - i)(\lambda + i)^{-1}$$

and

$$X_1 := (\text{Im} G)^{-1/2}(I - K_1)^{-1}(I - K_1^*K_1^*)^{1/2},$$

$$X_2 := (I - K_1^*K_1)^{-1/2}(I - K_1^*)^{-1}(\text{Im} G)^{1/2},$$

$$|\det X_1| |\det X_2| = 1.$$

It is known [24, 32] that the inner matrix-valued function $\tilde{S}_G(\lambda)$ is a Blaschke–Potapov product if and only if $\det \tilde{S}_G(\lambda)$ is a Blaschke product. From Lemma 5, the characteristic function $\tilde{S}_G(\lambda)$ is a Blaschke–Potapov product if and only if the matrix-valued function

$$X_K(\xi) = (I - K_1K_1^*)^{1/2}(\Theta(\xi) - K_1)(I - K_1^*\Theta(\xi))^{-1}(I - K_1K_1^*)^{1/2}$$

is a Blaschke–Potapov product in the unit disc.

**Definition 2.** Let $\tilde{E}$ be an $m$-dimensional $(m < \infty)$. In $\tilde{E}$ we set an orthonormal basis $e_1, e_2, \ldots, e_m$ and denote by $E_k$ $(k = 1, 2, \ldots, m)$ the linear span of vectors $e_1, e_2, \ldots, e_k$. If $M \subset E_k$, then the population of $x \in E_{k-1}$ with the property

$$\text{Cap} \left\{ \lambda : \lambda \in \mathbb{C}, \ (x + \lambda e_k) \subset M \right\} > 0$$

will be shown by $\Gamma_{k-1} M$ ($\text{Cap} G$ is the inner logarithmic capacity of a set $G \subset \mathbb{C}$). The $\Gamma$-capacity of a set $M \subset \tilde{E}$ is a number

$$\Gamma - \text{Cap} M := \sup \text{Cap}\{\lambda : \lambda e_1 \subset \Gamma_1 \Gamma_2 \ldots \Gamma_{m-1} M\},$$

where supremum is taken with respect to all orthonormal basis in $\tilde{E}$ (see [32,33]).

It is known that every set $M \subset \tilde{E}$ of zero $\Gamma$-capacity has zero $2m^2$-dimensional Lebesgue measure, however the converse of this case is not true [7].

Denote by $[\mathbb{C}^2]$ the set of all linear operators in $\mathbb{C}^2$. To convert $[\mathbb{C}^2]$ into an $n^2$-dimensional Hilbert space, we give the inner product $\langle T, S \rangle = \text{tr} S^*T$ for $T, S \in [E]$ ($\text{tr} S^*T$ is the trace of the operators $S^*T$). Hence we may give the $\Gamma$-capacity of a set in $\mathbb{C}^2$.

We use the following result of [32].
Lemma 6. Let \( X(\xi)(|\xi| < 1) \) be an analytic function with the values to be contractive operators in \([E]\) \((\|X(\xi)\| \leq 1)\). Then for \( \Gamma \)-quasi-every strictly contractive operators (i.e., for all strictly contractive \( K \in [C^2] \) possible with the exception of a set of \( \Gamma \) of zero capacity) the inner part of the contractive function

\[
X_K(\xi) = (I - K_1 K_1^*)^{1/2}(X(\xi) - K_1) (I - K_1^* X(\xi))^{-1}(I - K_1 K_1^*)^{1/2}
\]

is a Blaschke–Potapov product.

By summing all obtained result for the dissipative operator \( L_K(L_G) \), we have proved the following theorem.

Theorem 6. For \( \Gamma \)-quasi-every strictly contractive \( K \in [C^2] \) the characteristic function \( \tilde{S}_G(\lambda) \) of the dissipative operator \( L_K \) is a Blaschke–Potapov product and spectrum of \( L_K \) is purely discrete and belongs to the open upper half-plane. For \( \Gamma \)-quasi-every strictly contractive \( K \in [C^2] \) the operator \( L_K \) has an countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated vectors of this operators is complete in \( L_2(\mathbb{R}; C^2) \).

REFERENCES


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